SOME PROPERTIES OF THE DISTRIBUTION FUNCTION OF MAXWELL GAS PARTICLES

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The nature of the dependence of the distribution function on the initial conditions and of the changes with time in its high-frequency part are clarified.

In a number of cases it is important to know how the particle distribution function $f(t, \mathbf{r}, \mathbf{v})$ relaxes in the domain of large values of $|\mathbf{v}|$. Let us say that if inelastic processes are possible, their influence on the macroscopic gas parameters will differ depending on what fraction of the molecules has an energy comparable with the threshold of a given scattering channel. It is sometimes asserted that this part of the function f changes more rapidly than the rest. For example, the evolution of the initial distribution

$$f(0,\ \mathbf{r},\ \mathbf{v}) \equiv c \exp\{-b|\mathbf{v}|^2\}\, h(\mathbf{v}),\quad h(\mathbf{v}) \equiv \begin{cases} 1, & |\mathbf{v}| \leqslant v_0 = \text{const}, \\ 0, & |\mathbf{v}| > v_0 \end{cases} \tag{1}$$

of a gas of elastic spheres (more accurately, elastic circles since the model is two-dimensional) was considered in [1]. On the basis of computations performed using a calculating machine, it is concluded that the domain $|\mathbf{v}| > v_0$ is filled (Maxwellized) during a time on the order of the mean free path time τ_λ. The value of such results is apparently limited since the behavior of the function f is related to its asymptotic in this case. The asymptotic (in $|\mathbf{v}|$) properties of the distribution are studied by sufficiently rigorous methods herein.

Let there be a gas whose particles interact by means of a field with a Maxwell potential of bounded radius

$$\varphi(r_{ij}) \begin{cases} \sim r_{ij}^{-4}, & r_{ij} \leqslant r_0 = \text{const} > 0, \\ \equiv 0, & r_{ij} > r_0 \end{cases} \tag{2}$$

(r_{ij} is the spacing between the i-th and j-th particles, $i \neq j$). Let us examine just such initial distributions which are representable as

$$f(0,\ \mathbf{r},\ \mathbf{v}) \equiv g_0(\mathbf{r},\ \mathbf{v}) \exp\{-b|\mathbf{v}|^2\}, \tag{3}$$

where $|g_0(\mathbf{r}, \mathbf{v})| < \infty$, $g_0(\mathbf{r}, \mathbf{v})$ is a continuous function, $b = \text{const} > 0$. Moreover, for negligible simplifications we assume that there are no external fields (see [2]). Such a gas is described by the equation

$$Df \equiv \left(\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right) f = \int K\,(f'\underset{*}{f}' - f\underset{*}{f})\, d\omega\, d\underset{*}{\mathbf{v}} \tag{4}$$

($\mathbf{v}$, $\underset{*}{\mathbf{v}}$ are the particle velocities prior to the collision which results in the appearance of the velocities $\mathbf{v}'$, $\underset{*}{\mathbf{v}}'$; $f \equiv f(t, \mathbf{r}, \mathbf{v})$; $\underset{*}{f} \equiv f(t, \mathbf{r}, \underset{*}{\mathbf{v}})$, $f' \equiv f(t, \mathbf{r}, \mathbf{v}')$, $\underset{*}{f}' \equiv f(t, \mathbf{r}, \underset{*}{\mathbf{v}}')$; $K \equiv \sigma|\mathbf{v}-\underset{*}{\mathbf{v}}| \sin\vartheta$ is the kernel of the collision operator, $\sigma \equiv \sigma(|\mathbf{v}-\underset{*}{\mathbf{v}}|, \vartheta, \varphi)$ is the differential scattering cross section at the angles ϑ, φ; $d\omega \equiv d\vartheta d\varphi$; $d\underset{*}{\mathbf{v}} \equiv d\underset{*}{v}_1 d\underset{*}{v}_2 d\underset{*}{v}_3$; $\underset{*}{v}_k$ $(k = 1, 2, 3)$ are components of the vector $\underset{*}{\mathbf{v}}$).

Let us substitute the unknown function

$$f(t,\ \mathbf{r},\ \mathbf{v}) \equiv g(t,\ \mathbf{r},\ \mathbf{v}) \exp\{-b|\mathbf{v}|^2\} \tag{5}$$

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and let us go from the Cauchy problem (3)-(4) to its equivalent integral equation (see [3], [4])

$$g+\int_0^t \left[\int K_e(g\underset{*}{g}-g'\underset{*}{g}')\,d\omega d\underset{*}{\mathbf{v}}\right]_{t-\tau} d\tau-[g_0]_t\equiv B(g,\ g_0)=0, \tag{6}$$

$$K_e\equiv K\exp\{-b|\underset{*}{\mathbf{v}}|^2\}.$$

The braces $[\ ]_s$ are here the translation operator in the characteristics of the equation $Dw = 0$ ([3-5]). Let $t \in (0, t_1)$, $t_1 < \infty$, and let C denote the Banach space of bounded continuous functions dependent on t, $\mathbf{r}$, $\mathbf{v}$ ($t \in (0, t_1)$, $|\mathbf{r}| \le \infty$, $|\mathbf{v}| \le \infty$) with the norm of the element

$$\|x\|=\max_{t\mathbf{r},\mathbf{v}}|x|,\quad x\in C.$$

By virtue of Proposition (2), the inequality

$$k\equiv\int K_e d\omega d\underset{*}{\mathbf{v}}<\infty, \tag{7}$$

is satisfied. Hence, the operator $B(u, x)$ transforms $C \times C$ into C; moreover, it is continuous at any point $(u, x) \in C \times C$ and has a continuous (Frechet) derivative there equal to

$$B_u'(u,\ x)\,h\equiv h+\int_0^t\left[\int K_e(u\underset{*}{h}+\underset{*}{u}h-u'\underset{*}{h}'-\underset{*}{u}'h')\,d\omega d\underset{*}{\mathbf{v}}\right]_{t-\tau} d\tau. \tag{8}$$

The equation $B_u'(u, x)h = s$, or equivalently

$$h=\int_0^t\left[\int K_e(u'\underset{*}{h}'+\underset{*}{u}'h'-u\underset{*}{h}-\underset{*}{u}h)\,d\omega d\underset{*}{\mathbf{v}}\right]_{t-\tau} d\tau+s\equiv Lh, \tag{9}$$

is always solvable in C. Indeed, it is easy to show that

$$\|L^n z_1-L^n z_2\|\leqslant l_n\|z_1-z_2\|,\quad z_1,\ z_2\in C,$$
$$l_n\equiv\frac{(4\|u\|\,t_1 k)^n}{n!}, \tag{10}$$

and hence for sufficiently large n

$$l_n<1,$$

i.e., (9) has a unique solution in C (see [6]. But the element $s \in C$ is arbitrary; therefore, there exists an operator $(B_u'(u, x))^{-1}$. Finally, if $x_0 \equiv \gamma\exp\{-\beta|\mathbf{v}|^2\}$, β, γ = const, $\beta \ge 0$, then

$$B(x_0,\ x_0)\equiv 0. \tag{11}$$

The listed properties of the operators $B(u, x)$ and $B_u'(u, x)$ assure validity of the theorem ([7]).

<u>THEOREM</u>. There exist positive numbers δ_0 and δ such that if $\|x-x_0\| \le \delta_0$, then (6) has a unique solution in the neighborhood of $\|u-x_0\| \le \delta$.

The numbers δ_0 and δ are related to the quantity t_1 (the smaller the t_1, the greater they are) and to each other, namely

$$\delta\underset{\delta_0\to 0}{\to}0. \tag{12}$$

Therefore, the following can be asserted.

1. There exists a time interval within which the distribution function $f(t, \mathbf{r}, \mathbf{v})$ decreases no more slowly than the initial function $c|\mathbf{v}| \to \infty$ as $f(0, \mathbf{r}, \mathbf{v})$. The quantity t_1 depends on the properties of the distribution $f(0, \mathbf{r}, \mathbf{v})$; if δ_0 is sufficiently small, the inequality $t_1 \gg \tau_\lambda$ is satisfied, which does not agree with the deductions in [1].

2. The relation (12) shows that the solution of (6) depends continuously on g_0.

3. Since (11) is satisfied even for $\gamma \le 0$, the solution of (6) exists even if the function g_0 is sign-variable or negative. This latter property apparently is of no great value to the theory of the classical Boltzmann equation but can turn out to be essential for quantum theory (which admits of passage to an integral equation greatly analogous to (6), see [8]) since the Wigner one-particle function is not positive-definite.

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